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MARKOVIAN DECISION MODELS FOR THE EVALUATION OF A LARGE CLASS OF CONTINUOUS SAMPLING INSPECTION PLANS

Leon S. White

February, 1965 No. 104-64

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MARKOVIAN DECISION MODELS FOR THE EVALUATION OF A LARGE CLASS OF CONTINUOUS SAMPLING INSPECTION PLANS[1]

by

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1. Introduction and Summary. The purpose of this article is to present a uniform method for the evaluation of a large class S_D of Dodge-type continuous sampling inspection plans. The class of Dodge-type plans includes, among others, CSP-1, 2, 3, 4, and 5, MLP-1, r, and T, and H-106 plans. The evaluation of any plan $S \varepsilon S_D$ is in terms of its average outgoing quality limit (AOQL). The AOQL for S may be defined as an upper bound to the long run proportion of defective items that remains in the output after inspection, given certain assumptions about Nature's (the processes') ability to control process quality. In particular, a specific method of evaluation involving linear programming as its computational tool is developed for the case where Nature is assumed to be unrestricted in her ability to produce and submit defectives. The problem of determining unrestricted AOQL's for the plans in S_D is viewed in terms of two Markovian decisions models where Nature is taken to be the decision maker. These decision models are abstractly described in section 2. Their relation to the problem of evaluating continuous sampling plans is specified in section 3. The linear programs corresponding to the two decision models are derived in sections 4 and 5. In section 6 the linear programming approach is illustrated with an example,

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and in the last section a row reduction theorem is given for one of the linear programs. In addition, an appendix is included where a complete description of the class of Dodge-type continuous sampling inspection plans is given.

2. The Markovian Decision Models. Consider the following dynamic system as in Derman [4], [5], [6]. At times $t=0,1,\ldots$ the system is observed to be in some state i $(i=1,\ldots,L)$. After each observation the decision maker "controls" the system by making a decision d_k $(k=0,\ldots,K_i)$, where $K_i<\infty$ denotes the number of available decisions when the observed state is i. Let $\{Y_t: t=0,1,\ldots\}$ denote the sequence of observed states and $\{\Delta_t: t=0,1,\ldots\}$ the sequence of observed decisions. We shall assume that

$$P\{Y_{t+1}=j \mid h_{t-1}, Y_t=i, \Delta_t=d_k\} = q_{ij}(k)$$

$i,j=1,\ldots,L$; $k=0,\ldots,K_i$; $t=0,1,\ldots$, where for each t, h_t denotes the history of states and decisions (i.e., $h_t=\{Y_0,\Delta_0,\ldots,Y_t,\Delta_t\}$) and where the $q_{ij}(k)$'s are non-negative numbers which satisfy the equations,

$$\sum_j q_{ij}(k) = 1, \quad i=1,\ldots,L;\ k=0,\ldots,K_i.$$

The most general decision procedure (strategy for the decision maker) is defined by the collection of functions

$$\{D_k(h_{t-1}, Y_t=i): i=1,\ldots,L;\ k=0,\ldots,K_i;\ t=0,1,\ldots\}$$

of observations and decisions such that

$$D_k(\cdot) \geq 0, \quad \text{for all } k$$

and

$$\sum_k D_k(\cdot) = 1.$$

The class of <u>all</u> such procedures will be denoted by C. Any rule $R \in C$ is used to control the system by setting

$$P\{\Delta_t = d_k | h_{t-1}, Y_t\} = D_k(h_{t-1}, Y_t),$$

for all k and every possible set of observations (h_{t-1}, Y_t), $t=0,1,\ldots$. Once the initial state probabilities $P\{Y_0=i\}$, $i=1,\ldots,L$, are given and a rule R is designated, it follows that the sequence $\{Y_t : t=0,1,\ldots\}$ can be described as a stochastic process with state space $\{1,\ldots,L\}$.

Let C' denote the class of decision procedures R such that

$$D_k(h_{t-1}, Y_t=i) = D_{ik}, \quad i=1,\ldots,L; \; k=0,\ldots,K_i,$$

independent of h_{t-1} and t. Then if $R \in C'$, the sequence $\{Y_t : t=0,1,\ldots\}$ is a Markov chain with stationary transition probabilities

$$P_{ij} = \sum_i q_{ij}(k) D_{ik}, \quad i,j=1,\ldots,L.$$

In addition, let C" be the subset of C' such that $D_{ik} = 0$ or 1. (Note that this class contains only a finite number of procedures.)

Continuing as in Derman [4], let $w_{jk}(t) \geq 0$, $j=1,\ldots,L$; $k=0,\ldots,K_j$; $t=0,1,\ldots$ be finite numbers denoting some function of the observed state and decision taken. It will be assumed that $w_{jk}(t) = w_{jk}$ independent of t. Furthermore, let the finite numbers $w'_{jk}(t)>0$, $j=1,\ldots,L$; $k=0,\ldots,K_j$; $t=0,1,\ldots$ be defined in the same way as the $w_{jk}(t)$'s.

For a fixed procedure $R \in C$, define

$$\tilde{W}_t = \sum_{jk}\sum w_{jk} z_{jk}(t), \quad \tilde{W}'_t = \sum_{jk}\sum w'_{jk} z_{jk}(t)$$

where

$$z_{jk}(t) = \begin{cases} 1 & \text{if } Y_t=j,\ \Delta_t=d_k \\ 0 & \text{otherwise.} \end{cases}$$

Also, for $Y_0=i$ and any rule $R \in C$, let $\tilde{\Phi}^R_T(i)$ be the vectors with components

$$\tilde{x}_{jk}(T) = (T+1)^{-1} \sum_{t=0}^{T} z_{jk}(t), \quad j=1,\ldots,L;\ k=0,\ldots,K_j.$$

We shall be primarily concerned with the functions

$$\tilde{Q}_R(i) = \limsup_{T\to\infty} (T+1)^{-1} \sum_{t=0}^{T} \tilde{W}_t = \limsup_{T\to\infty} \sum_{jk}\sum w_{jk}\tilde{x}_{jk}(T)$$

and

$$\tilde{\psi}_R(i) = \limsup_{T\to\infty} \frac{\sum_{t=0}^{T} \tilde{W}_t}{\sum_{t=0}^{T} \tilde{W}'_t} = \limsup_{T\to\infty} \frac{\sum_{jk}\sum w_{jk}\tilde{x}_{jk}(T)}{\sum_{jk}\sum w'_{jk}\tilde{x}_{jk}(T)}.$$

Corresponding to these functions, we define $A_1(R,i)$ to be the smallest number A_1 such that

$$P\{\tilde{Q}_R(i) \leq A_1\} = 1,$$

and $A_2(R,i)$ to be the smallest number A_2 such that

$$P\{\tilde{\psi}_R(i) \leq A_2\} = 1.$$

We can now state the two problems that will be of interest in the sequel.

Problem 1. Suppose $P\{Y_0=i\} = 1$. Find a procedure R_1 such that

$$P\{\tilde{Q}_{R_1}(i) = \sup_{R\in C} A_1(R,i)\} = 1.$$

Problem 2. Suppose $P\{Y_0=i\} = 1$. Find a procedure R_2 such that

$$P\{\tilde{\psi}_{R_2}(i) = \sup_{R\in C} A_2(R,i)\} = 1.$$

3. Problems 1 and 2 and The Evaluation of Continuous Sampling Plans. Our interest in these problems arose in connection with a study of a class S_D of Dodge-type continuous sampling inspection plans and their evaluation, White [20]. Dodge-type continuous sampling plans [8], [9], [10], [11], [17], [19], [20], (hereafter abbreviated as DCSP's) are used to aid in the control of the percent defective in the output from a continuous production process. A typical DCSP allows for a mixture of 100% inspection and partial inspection

(e.g., if quality is good - no defectives have been found for some time - then partial inspection is employed; if quality is bad, then 100% inspection is used). The two most common methods of partial inspection are probability sampling where successive items are inspected with probability (say) B, and block sampling where one item is chosen at random for inspection from consecutive blocks of length (say) $\overline{B}$.

Many plans include several levels of partial inspection as well as one or more 100% levels. In general, the n^{th} level ($n=1,\ldots,\lambda+1$; $\lambda<\infty$) of a DCSP with probability sampling is defined by the pair (B_n,s_n) where B_n is the sampling probability and $s_n(s_n=1,\ldots,<\infty)$ is the release number for level n. The release number for $n \neq \lambda$ specifies the number of consecutive non-defectives that must be observed at level n in order to reduce the sampling frequency, i.e., switch to a "higher" sampling level. For level λ with $B_\lambda \leq B_n$, $n \neq \lambda$, it is mathematically convenient to let $s_\lambda = 1$ and to consider the switch in levels as fictitious, from λ to λ. The n^{th} level of a DSCP with block sampling is defined by the pair $(\overline{B}_n,s_n)$ where $\overline{B}_n$ is the block length associated with level n and s_n is the release number as defined before. In this case it is assumed that the levels are ordered such that $\overline{B}_\lambda \geq \overline{B}_n$, $n \neq \lambda$, and again $s_\lambda = 1$. For both probability and block sampling plans it is assumed that the 1^{st} level is a 100% inspection level, i.e., $B_1 = \overline{B}_1 = 1$.

At any time t, or now more precisely, at inspection point t ($t=0,1,\ldots$) the inspector can specify the state of his plan in terms of the sampling level and the number of consecutive non-defectives that he has observed while at that level. Therefore, the states of a DCSP can be defined by the set

$$N_U = \{n_u: \quad u=0,\ldots,s_n-1;\ n=1,\ldots,\lambda+1\}.$$

However, to be consistent with the state space notation previously introduced, we shall usually let

$$N_U = \{i: i=1,\ldots,L\},$$

where the relationship between i and n_u is given by

$$i = \sum_{v=1}^{n-1} s_v + u + 1.$$

A DCSP is often evaluated by its average outgoing quality limit (AOQL) relative to an assumed production process quality. A precise definition of the AOQL for any DCSP is most easily accomplished in three steps. First, outgoing quality (OQ_T) is defined as the proportion of defectives that remains in the total output after the $(T+1)^{st}$ inspection. In general, the OQ_T is a random variable. Second, the average outgoing quality (AOQ) is defined as the smallest number A such that

$$P\{\limsup_{T\to\infty} OQ_T \leq A\} = 1.$$

The AOQ can be thought of as a function of the inspector's (fixed) plan and the sequence of defectives and non-defectives "submitted" to the inspector. Suppose now that a possibly malevolent Nature determines this sequence. Assuming that Nature knows the inspector's plan prior to the start of production and that Nature can follow the progress of the plan once inspection begins, we can characterize the class of submission decision procedures available to Nature by

the class C introduced in Section 1. Then for a procedure RϵC, the types of decisions $\{d_k : k=0,\ldots,K_i\}$ available to Nature when the plan "is in state i" can be seen to depend on the state and on the method of sampling. Specifically, for a DCSP with probability sampling, $K_i = 1$, $i=1,\ldots,L$, and d_0 is interpreted as the decision to submit a non-defective, d_1 the decision to submit a defective. For a DCSP with block sampling, $K_i = \overline{B}_i$ and d_k corresponds to the decision to submit k defectives. Moreover, the sequence of decisions $\{\Delta_0,\Delta_1,\ldots\}$ which results from Nature's choice of a particular plan RϵC has the effect of controlling the inspector's sampling frequency. This can be seen by noting that if the plan is in state i at inspection point t and $\Delta_t=d_k$, then the next state (at inspection point t+1) is determined according to the probability vector $(q_{i1}(k),q_{i2}(k),\ldots q_{iL}(k))$. Therefore, if the AOQ is written as a function of R and fixed DCSP plan $S\epsilon S_D$, then the average outgoing quality limit for S is defined as

$$AOQL(S) = \sup_{R\epsilon C} AOQ(R,S).$$

Under this definition, Nature is virtually unrestricted in her choice of a decision procedure. Consequently, to differentiate this AOQL from others made under different assumptions about Nature, we shall refer to this AOQL as unrestricted and use the abbreviation UAOQL.

The connection between Problems 1 and 2 and the evaluation of DCSP's can now be made. For Problem 1 define

$$OQ_T = (T+1)^{-1} \sum_{t=0}^{T} \tilde{W}_t$$

and let the initial state $Y_0 = l_0$ with probability 1. (Actually, it is assumed that all DSCP's begin with 100% inspection.) Then for any DCSP with probability sampling $S_p \varepsilon S_D$, the submission decision procedure R_1 is such that

$$\text{UAOQL}(S_p) = A_1(R_1, l_0). \tag{3.1}$$

Similarly, for Problem 2 define

$$OQ_T = \frac{\sum_{t=0}^{T} \tilde{W}_t}{\sum_{t=0}^{T} \tilde{W}'_t}$$

and assume that $P\{Y_0 = l_0\} = 1$. Then for any DCSP with block sampling $S_b \varepsilon S_D$, the submission decision procedure R_2 is such that

$$\text{UAOQL}(S_b) = A_2(R_2, l_0). \tag{3.2}$$

Various methods for determining the UAOQL's of some DCSP's have been derived by Lieberman [16], Derman, et. al. [8], and Elfving [11]. Our results, presented in the next two sections, make it possible to use linear programming to evaluate UAOQL's for the class S_D of Dodge-type plans described in the Appendix.

Previous to this study, Derman [4], [5], has considered problems similar to Problems 1 and 2. In [4] and [5] the functions $\tilde{W}_t$ and $\tilde{W}'_t$ are defined as expectations rather than as random variables. As might be expected this difference is not material. In fact, in both studies it is shown that (i) attention may be restricted to the class C" in looking for optimal rules, and (ii) the problems can be solved using linear programming techniques.

4. Reduction to C".

Assumption A: For any initial state i(i=1,...,L) and any other state j≠i, there exists a rule R(i,j)εC" such that

$$P\{Y_t=j \text{ for some } t>0 \mid Y_0=i\} = 1$$

when R(i,j) is used.

Theorem 1. When assumption A holds, there exist procedures R_1 and R_2 in C" such that for Problem 1,

$$\tilde{Q}_{R_1}(i) = \max_{R\varepsilon C} A_1(R,i), \quad i=1,\ldots,L \tag{4.1}$$

with probability 1, and for Problem 2,

$$\tilde{\psi}_{R_2}(i) = \max_{R\varepsilon C} A_2(R,i), \quad i=1,\ldots,L \tag{4.2}$$

with probability 1.

Proof: The proof of both parts of the theorem depends on several results of Derman [6]. A statement of these results and the development of the proof require the following additional definitions. For every RεC and $Y_0=i$, let $\Phi_T^R(i)$ be the vectors with components

$$x_{jk}(T) = (T+1)^{-1} \sum_{t=0}^{T} P\{Y_t=j, \Delta_t=d_k \mid Y_0=i\}.$$

Let $\Phi^R(i) = \lim_{T\to\infty} \Phi_T^R(i)$ whenever the limit exists. This will occur when RεC' (See Chung [2], p. 32). In any case, let $H_R(i)$ denote the set of limit points

of $\{\phi_T^R(i)\}$ as $T\to\infty$, and let

$$H''(i) = \bigcup_{R\in C''} H_R(i).$$

For any $R\in C$, let ω denote a sample sequence of the joint process $\{(Y_t,\Delta_t): t=0,1,\ldots\}$ and define $\tilde{U}^R(i,\omega)$ to be the set of limit points of $\{\tilde{\phi}_T^R(i)\}$ as $T\to\infty$. Finally, let

$$g(\tilde{\phi}_T^R(i)) = (T+1)^{-1}\sum_{t=0}^{T}\tilde{W}_t, \qquad G(\tilde{\phi}_T^R(i)) = \frac{\sum_{t=0}^{T}\tilde{W}_t}{\sum_{t=0}^{T}\tilde{W}'_t},$$

and,

$$g(\phi_T^R(i)) = \sum\sum_{jk} w_{jk}x_{jk}(T), \qquad G(\phi_T^R(i)) = \frac{\sum\sum_{jk} w_{jk}x_{jk}(T)}{\sum\sum_{jk} w'_{jk}x_{jk}(T)}.$$

Lemma 1. (Derman [6]). There exists a procedure $R^*\in C''$ and an initial state i^* such that for any $R\in C$,

$$\tilde{Q}_R(i) = \operatorname{limsup}_{T\to\infty} g(\tilde{\phi}_T^R(i)) \tag{4.3}$$

$$\leq g(\phi^{R^*}(i^*))$$

with probability 1, where

$$g(\phi^{R^*}(i^*)) = \max_{R\in C;\ i=1,\ldots,L} \operatorname{limsup}_{T\to\infty} g(\phi_T^R(i)).$$

Lemma 2. (Theorem 2 of [6]). For any procedure $R \varepsilon C$ and initial state $Y_0 = i$,

$$P\{\tilde{U}^R(i,\omega) \varepsilon \overline{H}''\} = 1$$

where $\overline{H}''$ is the convex hull of $\bigcup_i H''(i)$.

Lemma 1 guarantees that $\tilde{Q}_R(i)$ is bounded from above, while Lemma 2 insures that all the limit points of any sequence $\{\tilde{\Phi}_T^R(i)\}$ lie in the closed and bounded convex set $\overline{H}''$. But now note that $\Phi^{R^*}(i^*) \varepsilon \overline{H}''$ and is in fact an extreme point of this set. Consequently, if we can construct a procedure $R_1 \varepsilon C''$ which leads to equality in (4.3) we then have a proof for the first half of Theorem 1.

To construct R_1 we assume without loss of generality that i^* is a member of an ergodic class (there may be more than one) of the Markov chain $\{Y_t: t=0,1,\ldots\}$ generated by R^*. Then for $Y_0 = i$ we assert that $R_1 = [R(i,i^*),R^*]$ where R_1 is interpreted as "use procedure $R(i,i^*)$ until state i^* is first reached, then use procedure R^*." By assumption A, $R_1 \varepsilon C''$. To show formally that R_1 leads to equality in (4.3) we prove the following lemma using some well known results from renewal theory.

Lemma 3. Suppose $Y_0 = i$ and that assumption A holds. Then if $R_1 = [R(i,i^*),R^*]$ it follows that

$$\tilde{Q}_{R_1}(i) = g(\Phi^{R^*}(i^*))$$

with probability 1.

Proof: Let $V_0 = \min\{t: Y_t=i^*\}$ given that $Y_0 = i$ and rule R_1 is being used. Define a cycle for the joint process $\{(Y_t,\Delta_t): t=0,.,...\}$ to be any sequence of the form $(Y_{\tau_1}=i^*,\Delta_{\tau_1}),...(Y_{\tau_2}=i^*,\Delta_{\tau_2})$ with $Y_t \neq i^*$, $\tau_1<t<\tau_2$. Let $V_r = (\tau_2 - \tau_1)$ for the r^{th} cycle, r=1,2,...; then $EV_r < \infty$ for any r. Let $z_r(j,k)$ denote the number of occurrences of the pair (j,k) during the r^{th} cycle, and let $z_0(j,k)$ denote the number of occurrences of this pair before i^* is first reached.

Now consider the sequence of points $\{T_a: a=0,1,...\}$ (dependent on ω) such that

$$\tilde{Q}_{R_1}(i) = \lim_{a\to\infty} g(\tilde{\Phi}^{R_1}_{T_a}(i)).$$

For any value of a it is easily seen that there exists a number M, either zero or a positive integer, such that

$$\sum_{r=0}^{M-1} V_r \leq T_a \leq \sum_{r=0}^{M} V_r - 1,$$

and such that

$$(4.4) \qquad \frac{\sum_{r=0}^{M-1} z_r(j,k)}{\sum_{r=0}^{M} V_r} < \tilde{x}_{jk}(T_a) < \frac{\sum_{r=0}^{M} z_r(j,k)}{\sum_{r=0}^{M-1} V_r - 1}$$

where $\tilde{x}_{jk}(T_a)$ is an element of $\tilde{\Phi}^{R_1}_{T_a}(i)$. The sequences $\{z_r(j,k): r=1,2,...\}$ and $\{V_r: r=1,2,...\}$ are both sequences of independent and identically distributed

random variables. Therefore, if the numerators and denominators of (4.4) are divided by M, then, by the strong law of large numbers,

$$\lim_{a\to\infty} \tilde{x}_{jk}(T_a) = x^*_{jk}$$

with probability 1, where x^*_{jk} is an element of $\Phi^{R^*}(i^*)$. And since the same result holds for all pairs (j,k), it follows from Lemmas 1 and 2 and the continuity of g that

$$\tilde{Q}_{R_1}(i) = g(\Phi^{R^*}(i^*))$$

with probability 1.

Combining lemmas 1, 2 and 3 we have that

$$\tilde{Q}_{R_1}(i) = \max_{R\in C} A_1(R,i)$$

with probability 1 which proves the first half of the theorem.

To prove the second half of Theorem 1 we shall need the following well known proposition.

Lemma 4. Let H be a closed and bounded convex set. Let g(x) and g'(x) be linear functions defined on H with g'(x)>0 for all $x\in H$. Then the ratio G(x) = g(x)/g'(x) takes its maximum at an extreme point of H.

Lemma 4 is used in proving the next lemma which is the counterpart to lemma 1.

Lemma 5. There exists a procedure $R^{**}\varepsilon C''$ and an initial state i^{**} (again without loss of generality, a member of an ergodic class) such that for any $R\varepsilon C$,

$$\tilde{\psi}_R(i) = \limsup_{T\to\infty} G(\tilde{\phi}_T^R(i)) \leq G(\phi^{R^{**}}(i^{**})) \tag{4.5}$$

with probability 1, where

$$G(\phi^{R^{**}}(i^{**})) = \max_{R\varepsilon C;\ i=1,\ldots,L} G(X^R(i)), \tag{4.6}$$

$X^R(i)\varepsilon H_R(i)$, $X^R(i)$ denoting any limit point of any sequence $\{\phi_T^R(i)\}$.

Proof: Consider the closed and bounded set $\bar{H}''$. By a second theorem of Derman [6], $X^R(i)\varepsilon\bar{H}''$ for all $R\varepsilon C$ and $i=1,\ldots,L$. But then by Lemma 4 and the construction of $\bar{H}''$ it follows that there exists a procedure $R^{**}\varepsilon C''$ an initial state i^{**} such that the ratio function G is maximized at the extreme point $\phi^{R^{**}}(i^{**})$. Equation (4.6) denotes this fact. Now consider the sequence of points $\{T_a: a=0,1,\ldots\}$ (dependent on the sample function ω) such that

$$\limsup_{T\to\infty} G(\tilde{\phi}_T^R(i)) = \lim_{a\to\infty} G(\tilde{\phi}_{T_a}^R(i)).$$

Next, recall from Lemma 2 that for some subsequence $\{T_a(\alpha): \alpha=0,1,\ldots\}$, $\lim_{a\to\infty} \tilde{\phi}_{T_a}^R(\alpha)(i)\varepsilon\bar{H}''$ with probability 1. It follows that $\tilde{\psi}_R(i)$ can be considered as a function on $\bar{H}''$. And since G is a continuous function on $\bar{H}''$, (4.5) follows and the proof of the lemma is complete.

Now we define the procedure $R_2 = [R(i,i^{**}),R^{**}]$ and again by assumption A $R_2 \epsilon C''$. Then similar to Lemma 3 we have

Lemma 6. Suppose $Y_0 = i$ and assumption A holds. Then if $R_2 = [R(i,i^{**}),R^{**}]$ it follows that

$$\tilde{\psi}_{R_2}(i) = G(\Phi^{R^{**}}(i^{**}))$$

with probability 1.

Lemmas 4, 5, and 6 combine to prove that

$$\tilde{\psi}_{R_2}(i) = \max_{R \epsilon C} A_2(R,i)$$

with probability 1. This completes the proof of Theorem 1.

Corollary Let the elements of $\Phi^{R^*}(i^*)$ and $\Phi^{R^{**}}(i^{**})$ be denoted by $\{x^*_{jk}\}$ and $\{x^{**}_{jk}\}$ respectively. Then

$$A_1(R_1,i) = \sum\sum_{jk} w_{jk} x^*_{jk}, \quad i=1,\ldots,L, \tag{4.7}$$

and

$$A_2(R_2,i) = \frac{\sum\sum_{jk} w_{jk} x^{**}_{jk}}{\sum\sum_{jk} w'_{jk} x^{**}_{jk}}, \quad i=1,\ldots,L. \tag{4.8}$$

Proof: Equations (4.7) and (4.8) follow directly from Lemmas 3 and 6.

The implication of the theorem and corollary for the evaluation of DCSP's is clear. The interpretation of Theorem 1 is that a malevolent Nature facing any DCSP can choose her optimal submission decision procedure from the class C". Moreover, from the corollary and equations (3.1) and (3.2) it follows that the UAOQL for any DCSP is computed either as a linear function or as the ratio of two linear functions.

5. Linear Programming Formulations. The linear programming formulation of Problems 1 and 2 follows from the corollary to Theorem 1 and Markov chain considerations. The details of such a formulation are discussed in Derman [4], [7]. The method has also been used by Manne [18], Klein [14], [15] and Dantzig and Wolfe [3] among others. (Howard [13] has provided an alternative approach for similar problems.) In this section we state the problems as they apply specifically to the evaluation of DCSP's.

Problem 1. The UAOQL for any DCSP with probability sampling is given by the solution to the following problem:

$$\text{Maximize:} \quad \sum_{jk}\sum w_{jk}x_{jk}$$

subject to:

$$x_{jk} \geq 0, \qquad j=1,\ldots,L;\ k=0,1$$

(5.1)

$$\sum_{k} x_{jk} - \sum_{ik}\sum x_{ik}q_{ij}(k) = 0, \qquad j=1,\ldots,L$$

$$\sum_{ik}\sum x_{ik} = 1.$$

Nature's optimal procedure is found by setting

$$D_{jk} = \frac{x_{jk}}{\sum_{m} x_{jm}}, \qquad j=1,\ldots,L;\ k=0,1.$$

If for some state j the denominator is zero then $\{D_{jk}: k=0,1\}$ should be set so that j is a transient state. Nature can always accomplish this by setting $D_{j0} = 1$ or 0 depending on j.

In formulating Problem 2 we also make use of the fact that the ratio of two linear functions defined on a convex set can be maximized by linear programming techniques. (See Derman [4] or Charnes and Cooper [1] for details.)

Problem 2. The UAOQL for any DCSP with block sampling is given by the solution to the following problem:

$$\text{Maximize:} \quad \sum_{jk}\sum w_{jk}y_{jk}$$

subject to:

$$y_0 \geq 0,\ y_{jk} \geq 0, \qquad j=1,\ldots,L;\ k=0,\ldots,\bar{B}_j$$

$$\sum_{k} y_{jk} - \sum_{ik}\sum y_{ik}q_{ik}(k) = 0, \qquad j=1,\ldots,L$$

$$\sum_{jk}\sum y_{jk} - y_0 = 0$$

$$\sum_{jk}\sum w'_{jk}y_{jk} = 1.$$

As before, the elements of Nature's optimal rule can be computed from the formula,

$$D_{jk} = \frac{y_{jk}}{\sum_{m} y_{jm}}, \qquad j=1,\ldots,L;\ k=0,\ldots,\overline{B}_j.$$

6. An Example: The Evaluation of CSP-1 by Linear Programming. The first continuous sampling inspection plan, CSP-1, was devised by Dodge and reported in 1943 [10]. The object of this plan as stated by Dodge ([10], p. 264) is "to establish a limiting value of average outgoing quality expressed in percent defective which will not be exceeded no matter what quality is submitted to the inspector." The following is a description of CSP-1 with probability sampling:

(a) Begin by inspecting 100% of the output until s units in succession are found to be non-defective.

(b) When s successive non-defective units are produced switch from 100% inspection to partial inspection, inspecting each successive item with probability B.

(c) If a sample unit is found to be defective, revert immediately to 100% inspection and continue as in paragraph (a).

(d) Replace all defective items found with non-defectives.

Lieberman [16] using probability theoretic considerations derived a formula to compute the UAOQL for any CSP-1 plan with probability sampling. The linear programming approach leading to the same formula is as follows.

Let the state space for a CSP-1 plan be given by $N_U = \{1_0, 1_1, \ldots, 1_{s-1}, 3_0\}$ (See the appendix for details on notation.) Let $w_{n_u k}$ be defined as the expected number of defectives that pass by the inspector when the plan is in state n_u and Nature takes decision d_k. Then

$$w_{n_u k} = \begin{cases} (1-B), & n=3; u=0; k=1 \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

The conditional transition probabilities ($q_{ij}(k)$) for CSP-1 are given by,

$$q_{1_u n_v}(k) = \begin{cases} 1, & k=0;\ u=0,\ldots,s-2;\ n=1;\ v=u+1 \\ 1, & k=0;\ u=s-1\ n=3;\ v=0 \\ 1, & k=1;\ u=0,\ldots,s-1;\ n=1;\ v=0 \\ 0, & \text{otherwise,} \end{cases} \tag{6.2}$$

and

$$q_{3_0 n_v}(k) = \begin{cases} 1, & k=0;\ n=3;\ v=0 \\ 1-B, & k=1;\ n=3;\ v=0 \\ B, & k=1;\ n=1;\ v=0 \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Now by substituting (6.1), (6.2), and (6.3) into the linear programming formulation of Problem 1 we have for CSP-1 with probability sampling the linear program,

$$\text{Maximize:} \quad (1-B)x_{3_0 1}$$

subject to:

$$x_{n_u k} \geq 0, \qquad \text{all } n_u \epsilon N_U;\ k=0,1$$

$$x_{1_0 0} + x_{1_0 1} - \sum_{u=0}^{s-1} x_{1_u 1} - Bx_{3_0 1} = 0$$

$$x_{1_u 0} + x_{1_u 1} - x_{1_{u-1} 0} = 0, \quad u=1,\ldots,s-1 \tag{6.4}$$

$$\sum_{nuk}\sum\sum x_{n_u k} = 1.$$

Note: We have omitted one equation in (6.4) since in the original set of constraints (5.1), the sum of the first L equations equals the $(L+1)^{st}$.)

This program can be solved by the usual methods. However, because of the form of the objective function and the last constraint equation, any optimal solution is given in part by $x_{1_u 1} = 0$, $u=0,\ldots,s-1$, and $x_{3_0 0} = 0$. But then letting $x_{nk} = x_{n_u k}$, we can rewrite the above linear program as

$$\text{Maximize:} \quad (1-B)x_{31}$$

subject to:

$$x_{10} - Bx_{31} = 0$$

$$sx_{10} + x_{31} = 1$$

The solution to this program is easily seen to be,

$$x_{10} = \frac{B}{sB+1}, \qquad x_{31} = \frac{1}{sB+1}, \qquad x_{11} = x_{30} = 0,$$

and consequently,

$$UAOQL = \frac{1-B}{sB+1}.$$

It follows, then, that Nature's decision rule is given by $D_{1_{n^0}} = 1$, $u=0,\ldots,s-1$ and $D_{3_{0}1} = 1$.

Derman, et.al. [8] have shown that when $B = 1/\overline{B}$, the same decision rule is optimal against a CSP-1 block sampling plan with parameters $\overline{B}$ and s. This conclusion has also been arrived at by the linear programming approach to Problem 2 in White [20]. In the linear programming formulation, $w_{n_u k}$ is again defined as the expected number of defectives that pass by the inspector at inspection point t when $Y_t = n_u$ and $\Delta_t = d_k$. In addition, $w'_{n_u k}$ is defined as the expected number of items (both defective and non-defective) that pass by the inspector at inspection point t when $Y_t = n_u$ and $\Delta_t = d_k$. Thus, for a CSP-1 plan with block sampling,

$$w_{n_u k} = \begin{cases} 0, n=1;\ u=0,\ldots,s-1;\ k=0,1 \\ \\ k[1-(k/\overline{B})]+(k-1)(k/\overline{B}), n=3;\ u=0;\ k=0,\ldots,\overline{B} \end{cases}$$

and

$$w'_{n_u k} = \begin{cases} 1, & n=1;\ u=0,\ldots,s-1;\ k=0,1 \\ \overline{B}, & n=3;\ u=0;\ k=0,\ldots,\overline{B}. \end{cases}$$

The linear programming method of Problem 1 has also been used by White [20] to find UAOQL's for the class of H-106 plans [19].

7. A Row Reduction Theorem. In the example of section 6 it was shown that the original set of s+1 constraint equations could be replaced by a set of two equations. This result for a two level plan can be generalized for a $\lambda+1$ level DCSP with probability sampling.

Theorem 2. Let

$$\begin{aligned} x_{n_u 0} &= x_{n0}, && u=0,\ldots,s_n-1;\ n=1,\ldots,\lambda+1 \\ x_{n_0 1} &= x_{n1}, && n=1,\ldots,\lambda+1 \\ x_{n_u 1} &= 0, && u=1,\ldots,s_n-1;\ n=1,\ldots,\lambda+1 \end{aligned} \tag{7.1}$$

Then the linear program for Problem 1 can be written in the following reduced form:

$$\text{Maximize: } \sum_{n=1}^{\lambda+1} (1-B_n)x_{n1}$$

subject to:

$$x_{nk} \geq 0, \qquad n=1,\ldots,\lambda+1;\ k=0,1,$$

$$\sum_{k=0}^{1} x_{nk} - \sum_{m=1}^{\lambda+1} \sum_{k=0}^{1} x_{mk} q_{mn}(k) = 0, \qquad n=1,\ldots,\lambda+1,$$

$$\sum_{n=1}^{\lambda+1} (s_n x_{n0} + x_{n1}) = 1,$$

where,

$$q_{mn}(0) = \begin{cases} q_{m_{s-1}n_0}(0), & m \neq n \\ q_{n_0 n_0}(0), & m=n \end{cases}$$

and

$$q_{mn}(1) = q_{m_0 n_0}(1).$$

Proof: The proof of this theorem follows from the observation that if Nature is going to submit defectives at any level n (n=1,...,λ+1), then she can always begin submitting them at state n_0 without reducing (and possibly increasing) the value of the objective function. Thus, only the class of feasible solutions with this property need be considered. This class is described by the equations (7.1).

Appendix: Description of a Class of DCSP's. The class of all DCSP is denoted by S_D. Any plan in S_D is described by specifying the type of sampling during partial inspection, the set of sampling levels, an inspection rule and a transition rule. For descriptive purposes, the class S_D is divided into two sub-classes S_1 and S_2. S_1 consists of all plans that use probability sampling and S_2 includes all plans that use block sampling.

The levels of a typical plan $S \varepsilon S_D$ consist of an initial level at which inspection begins, basic levels, and possibly associated levels for some (or all) basic levels. If probability sampling is used, the inspection rule of S designates the probability of inspection at each level. If block sampling is used, the inspection rule specifies a block length and inspection-disposition procedure for each level. The inspection-disposition procedure for any level consists of a method of block inspection including a screening procedure if a defective is found. The transition rule of S controls the changes in the amount of inspection to be performed by the inspector. In general, if the inspector finds a defective, the transition rule causes the frequency of inspection to increase immediately. On the other hand, if a sufficient number of successive non-defectives are found, the transition rule causes a reduction in the inspection rate. The exact specification of the class of acceptable transition rules for S_1 and S_2 is given below.

The sub-class S_1: The n^{th} level $(n=1,\ldots,\lambda+1)$ of any sampling plan $S \varepsilon S_1$ is defined by the pair (B_n, s_n), where $B_n (0 < B_n \leq 1)$ is the probability of inspecting an item while at level n, and $s_n (s_n = 1, 2, \ldots, <\infty)$ is the release number

for level n. The release number for level $n \neq \lambda$ specifies the number of successive non-defectives that must be observed in order to switch sampling to a different level. For level λ, $s_\lambda = 1$ and the plan stays in level λ as long as non-defectives are observed.

Level 1, where inspection begins, is termed the initial level and $B_1 = 1$ for all $S \varepsilon S_1$. The basic levels are numbered $n=3,6,\ldots,\lambda(\lambda<\infty)$. Any basic level may have two associated levels. Associated levels for any basic level $n<\lambda$ are denoted by (B_{n-1},s_{n-1}) and (B_{n+1},s_{n+1}) where $B_{n-1} = 1$, $s_{n-1} < s_n$, $B_{n+1} = B_n$ and $s_{n+1} = (s_n - s_{n-1})$. Associated levels for level λ differ only in that no restrictions are placed on the release numbers $s_{\lambda-1}$ and $s_{\lambda+1}$. We shall let N(S) denote the set of level numbers of plan $S \varepsilon S_D$.

The inspection rule of any plan $S \varepsilon S_1$ is given by the set $\{B_n : n \varepsilon N(S)\}$. An inspection point is a point in time when the inspection rule is applied. At any inspection point $t(t=0,1,\ldots)$ the application of the inspection rule of any plan with probability sampling will lead to one of three results:

$$r = \begin{cases} 0, & \text{if the item is inspected and is non-defective} \\ 1, & \text{if the item is inspected and is defective} \\ 2, & \text{if the item is not inspected.} \end{cases}$$

It is assumed for all $S \varepsilon S_D$ that an inspected item is always correctly classified. For all $S \varepsilon S_1$ it is assumed that a defective, if discovered, is removed and replaced with a non-defective. The sequence of results is denoted by the set $\{\beta_t : t=0,1,\ldots\}$.

The state of a sampling plan $S \varepsilon S_D$ at any inspection point t is designated by n_u. The set of all states is denoted by

$$N_U = \{n_u : u=0,\ldots,s_n-1;\ n \varepsilon N(S)\}.$$

n_u denotes the fact that the plan is at level n and u consecutive non-defectives have been observed while at that level.

What ever the present state of a plan, the transition decision determines the next state with certainity. This decision is based on the present state n_u and the sampling result r. It is denoted by,

$$(1) \qquad e_{n_u m_v}(r) = \begin{cases} 1, \text{ for one } m_v \varepsilon N_U \\ \\ 0, \text{ for all other } m_v \varepsilon N_U. \end{cases}$$

Thus, $e_{n_u m_v}(r) = 1$ means that when $Y_t = n_u$ and $\beta_t = r$, then $Y_{t+1} = m_v, t=0,1,\ldots$. The relationship between transition decisions and the conditional transition probabilities ($q_{n_u m_v}(k)$) is given by

$$(2) \qquad q_{n_u m_v}(k) = \sum_{\nu} e_{n_u m_v}(\nu) P\{\beta_t = \nu | Y_t = n_u, \Delta_t = d_k\}$$

$n_u, m_v \varepsilon N_U$; $k=0,\ldots,K_{n_u}$; $t=0,1,\ldots$. Equation (2) holds for all plans $S \varepsilon S_D$.

A transition rule consists of a set of transition decisions, one for each pair (n_u, r), $n_u \varepsilon N_U$; $r=0,1,2$. A transition rule is represented by the set of matrices $\{E(r): r=0,1,2\}$, where the $(n_u, m_v)^{th}$ element of E(r) is given by (1).

In general terms, an acceptable transition rule is one that causes the inspection frequency to diminish if no defectives are observed during a

sufficient number of successive inspections, and one that immediately causes the inspection rate to increase if a defective is found.

A complete description of the set of acceptable transition rules is facilitated by putting an ordering over the levels of any probability sampling plan. This ordering is as follows: level n is said to be higher than level m (denote by $n \succ m$, $n \neq m$) if either $B_n < B_m$ or if $B_n = B_m$ but $s_n < s_m$; otherwise, n is lower than m. Furthermore, the ordering requires that the release numbers are such that

$$
\begin{aligned}
&s_1 > s_n, \text{ for all } n>1, \\
&s_{m-1} \geq s_{n-1}, \text{ if n is a higher basic level than m} \\
&s_{m+1} \geq s_{n+1}, \text{ if n is a higher basic level than m.}
\end{aligned} \tag{3}
$$

Thus if the initial level and basic levels are ordered numerically by the "higher than" criterion (i.e., level 1 is the lowest level and level λ is the highest) then with one exception if $j \succ i$ then $j>i$; the exception is that $\lambda \succ \lambda+1$. Assuming this numerical ordering scheme, we define the set of acceptable transition rules as those whose transition decisions meet the following restrictions:

(i) Within any level n except the initial level, (r-1) type associated levels, and level λ,

$$
e_{n_u n_v}(r) = \begin{cases} 1, & r=0;\ v=u+1;\ u=0,\ldots,s_n-2 \\ 1, & r=2;\ v=u;\ u=0,\ldots,s_n-1 \\ 0, & \text{otherwise;} \end{cases}
$$

within the initial level,

$$e_{1_u 1_v}(r) = \begin{cases} 1, & r=0;\ v=u+1;\ u=0,\ldots,s_1-2 \\ 1, & r=1;\ v=0;\ u=0,\ldots,s_1-1 \\ 0, & \text{otherwise;} \end{cases}$$

within any (n-1)-type associated level,

$$e_{(n-1)_u (n-1)_v}(r) = \begin{cases} 1, & r=0;\ v=u+1;\ u=0,\ldots,s_{(n-1)}-2 \\ 0, & \text{otherwise;} \end{cases}$$

and within level λ,

$$e_{\lambda\lambda}(r) = \begin{cases} 1, & r=0,2 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Between levels n and m, $n \neq m$, and when the present level $n = 3, 6, \ldots, \lambda-3$,

$$e_{n_u m_v}(r) = 0, \quad r=0,1,2;\ n\neq m;\ v\neq 0;\ u=0,\ldots,s_n-1;$$

$$e_{n_u m_1}(r) = \begin{cases} 1, & r=0;\ m=n+3;\ u=s_n-1 \\ 0, & r=0;\ m\neq n;\ u=0,\ldots,s_n-2 \\ 0, & r=0;\ m\neq n+3;\ u=s_n-1 \\ 1, & r=1;\ \text{one } m<n;\ u=0,\ldots,s_n-1 \\ 0, & r=1;\ \text{all other } m;\ u=0,\ldots,s_n-1 \\ 0, & r=2;\ m\neq n;\ u=0,\ldots,s_n-1 \end{cases}$$

and for the initial level,

$$e_{1_u m_v}(r) = \quad 0,\ r=0,1,2;\ m\neq 1;\ v\neq 0;\ u=0,\ldots,s_1-1$$

$$e_{1_u m_1}(r) = \begin{cases} 1,\ r=0;\ m=3;\ u=s_1-1 \\ 0,\ r=0;\ m\neq 1;\ u=0,\ldots,s_1-2 \\ 0,\ r=0;\ m\neq 3;\ u=s_1-1 \\ 1,\ r=1;\ m=1;\ u=0,\ldots,s_1-1 \\ 0,\ r=1;\ m\neq 1;\ u=0,\ldots,s_1-1 \\ 0,\ r=2;\ \text{all } m;\ u=0,\ldots,s_1-1; \end{cases}$$

(iii) Between levels when the present level is an associated level of basic level n,

$$e_{(n-1)_u m_v}(r) = \quad 0,\ r=0,1,2;\ m\neq n-1;\ v\neq 0;\ u=0,\ldots,s_n-1;$$

$$e_{(n-1)_u m_1}(r) = \begin{cases} 1,\ r=0;\ m=r+1;\ u=s_n-1 \\ 0,\ r=0;\ m\neq n+1;\ u=0,\ldots,s_{(n-1)}-2 \\ 0,\ r=0;\ m\neq n+1;\ u=s_{(n-1)}-1 \\ 1,\ r=1;\ \text{one } m<n;\ u=0,\ldots,s_{(n-1)}-1 \\ 0,\ r=1;\ \text{all other } m;\ u=0,\ldots,s_{(n-1)}-1 \\ 0,\ r=2;\ \text{all } m;\ u=0,\ldots,s_{(n-1)}-1 \end{cases}$$

and,

$$e_{(n+1)_u m_v}(r) = \quad 0,\ r=0,1,2;\ m\neq n+1;\ v\neq 0;\ u=0,\ldots,s_{(i+1)}-1;$$

$$e_{(n+1)_u m_1}(r) = \begin{cases} 1,\ r=0;\ m=n+3;\ n\neq\lambda;\ u=s_{(n+1)}-1 \\ 1,\ r=0;\ m=n=\lambda;\ u=s_{\lambda+1}-1 \\ 0,\ r=0;\ m\neq n+1;\ u=0,\ldots,s_{(n+1)}-2 \\ 0,\ r=0;\ m\neq n+3;\ u=s_{(n+1)}-1 \\ 1,\ r=1;\ \text{one } m<n;\ u=0,\ldots,s_{(n+1)}-1 \\ 0,\ r=1;\ \text{all other } m;\ u=0,\ldots,s_{(n+1)}-1 \\ 0,\ r=2;\ m\neq n+1;\ u=0,\ldots,s_{(n+1)}-1 \end{cases}$$

In words, these restrictions amount to the following. When a non-defective is observed at any level $n \neq \lambda$, a change of state follows. This change is to <u>stage</u> u+1 of level n if the present state $n_u \neq s_n-1$. If $n_u = s_n-1$, then the change is to a specific higher level. On the other hand, when a defective is found at any level, a change of state occurs in the other direction. Here the change is always to the first stage of a lower level except when the plan is in level 1; then the change is to state 1_0. When no inspection is performed and consequently no information is obtained, the restrictions insure that no change of state occurs.

It is now possible to uniquely specify any plan $S \varepsilon S_1$ as

$$S = [\{(B_n, s_n): u \varepsilon N(S)\}; \{E(r): r=0,1,2\}] \tag{4}$$

whenever the assumptions and restrictions given above are met.

<u>The sub-class S_2</u>: The n^{th} level ($n=1,\ldots,\lambda+1$) of any sampling plan $S \varepsilon S_2$ is defined by the pair $(\overline{B}_n, s_n)$, where $\overline{B}_n$ ($\overline{B}_n=1,2,\ldots,<\infty$) is the <u>block length</u> for level n and s_n is, as before, the release number. Also as before, the

initial level, level 1, is a 100% inspection level ($\bar{B} = 1$), the basic levels are numbered $n=3,6,\ldots,\lambda$, and each basic level may have two associated levels. An (n-1)-type associated level is defined by $(\bar{B}_{n-1}, s_{n-1})$ where $\bar{B}_{n-1} = 1$ and $s_{n-1} < s_n$ for $n<\lambda$; an (n+1)-type associated level is defined by $(\bar{B}_{n+1}, s_{n+1})$ where $\bar{B}_{n+1} = \bar{B}_n$ and $s_{n+1} = (s_n - s_{n-1})$, also for $n<\lambda$. Associated levels for level λ are given by $(1, s_{\lambda-1})$ and $(\bar{B}_\lambda, s_{\lambda+1})$ where the release numbers are arbitrary positive numbers.

An inspection-disposition procedure for each block inspected at any level $n \in N(S)$, $S \in S_2$, includes a method of inspection and a rule for disposing of the inspected and uninspected items in the block. The following are six possible inspection-disposition procedures:

$I_n = 1$: Inspect one item and classify it. If the item is defective replace it with a non-defective. (This procedure is mandatory when $\bar{B}_n = 1$; it is not used otherwise.)

$I_n = 2$: Inspect, at random, one item from the block. If it is defective replace it with a non-defective, but, regardless of its quality allow the remaining members of the block to pass by the inspection station.

$I_n = 3$: Inspect, at random, one item from the block. If it is defective replace it with a non-defective, but, do not let any of the remaining members of the block pass by the inspection station. (Used in Derman, et.al. [8].)

$I_n = 4$: Inspect, at random, one item from the block. If it is defective replace it with a non-defective and in addition inspect all the remaining members of the block allowing only non-defective items to pass. (Used in Derman, et.al. [8].)

$$r = \begin{cases} 0, \text{ if } I_n \text{ leads to the discovery of no defectives} \\ 1, \text{ if } I_n \text{ leads to the discovery of one defective} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \rho, \text{ if } I_n \text{ leads to the discovery of the maximum number of defectives that can be found using } I_n. \end{cases}$$

When inspection-disposition procedures 4, 5 or 6 are applied, the possible inspection results will vary more widely than with probability sampling. Consequently, the corresponding transition decision rules may be made more complex. However, in all cases, it is again assumed that a transition decision depends only on the present state of the plan and on the inspection results.

A transition decision that depends on the present state and on the outcome of the initial sample (of size one) from a block is called a single-step transition decision. The reason for doing additional inspection while using this transition decision, i.e., the reason for setting $I_n = 4, 5$ or 6, is simply to reduce the number of defectives in the output. However, if additional inspection is also done as a basis for making the transition decision, then the resulting decision is termed a two-step transition decision.

Clearly, there are many possible two-step rules. Of these we suggest the following one: if $r = 0$ or 1 use the equivalent of an ordinary single-step rule; if $r>1$, i.e., at least one defective has been found in the second sample, return the plan to state l_0. For each level n, denote the type of transition decision to be used by,

$$J_n = \begin{cases} 1, \text{ for a single-step transition decision} \\ 2, \text{ for a two-step transition decision.} \end{cases}$$

As before, for a given present state n_u and result r, the transition decision, whether single-step or two-step, determines the next state of the sampling plan with certainty. For a block sampling plan, it is represented by its type J_n and the set $\{e_{n_u m_v}(r): m_v \varepsilon N_U;\ r=0,\ldots,\bar{B}_\lambda\}$ where,

(i) if $I_n = 1$, 2 or 3 and $J_n = 1$, then

$$(5) \qquad e_{n_u m_v}(r) = \begin{cases} 1, \text{ for one } m_v \varepsilon N_U \text{ when } r \leq 1 \\ 0, \text{ for all other } m_v \varepsilon N_U \text{ when } r \leq 1 \\ 0, \text{ for all } m_v \varepsilon N_U \text{ when } r > 1; \end{cases}$$

(ii) if $I_n = 4$ or 5 and $J_n = 1$ or 2, then

$$(6) \qquad e_{n_u m_v}(r) = \begin{cases} 1, \text{ for one } m_v \varepsilon N_U \text{ when } r \leq \bar{B}_n \\ 0, \text{ for all other } m_v \varepsilon N_U \text{ when } r \leq \bar{B}_n \\ 0, \text{ for all } m_v \varepsilon N_U \text{ when } r > \bar{B}_n; \end{cases}$$

(iii) if $I_n = 6$ and $J_n = 1$ or 2, then

$$(7) \qquad e_{n_u m_v}(r) = \begin{cases} 1, \text{ for one } m_v \varepsilon N_U \text{ when } r \leq \min(a, \bar{B}_n) \\ 0, \text{ for all other } m_v \varepsilon N_U \text{ when } r \leq \min(a, \bar{B}_n) \\ 0, \text{ for all } m_v \varepsilon N_U \text{ when } r > \min(a, \bar{B}_n). \end{cases}$$

The transition rule of any plan $S \varepsilon S_2$ is then defined by the set $\{J_n;\ n \varepsilon N(S)\}$ and the set of the matrices $\{E(r):\ r=0,\ldots,\overline{B}_\lambda\}$ where the $(n_u, m_v)^{th}$ element of $E(r)$ is given by (5), (6) or (7) depending on the value of I_n.

To specify the set of acceptable transition rules for plans with block sampling, we first define an ordering on the levels of any such plan as follows. Level n is said to be higher than level m if $\overline{B}_n > \overline{B}_m$ or if $\overline{B}_n = \overline{B}_m$ but $s_n < s_m$; otherwise level n is lower than level m. Furthermore, it is again required that the inequalities of (3) hold. Under this ordering, the set of acceptable transition rules are those whose transition decisions meet the following restrictions:

(i) for single-step transition decision, the restrictions are the same as those given for probability sampling transition decisions when r = 0 or 1. The only additional restriction is that if $r>1$, then the restrictions for r = 1 are to apply.

(ii) for two-step transition decisions, the restrictions are again the same as those given for probability sampling transition decisions when r = 0 or 1. The only additional restriction is the following: when the present state is n_u and $r>1$, then according to the two-step rule previously suggested,

$$e_{n_u m_v}(r) = \begin{cases} 1 \text{ for } m_v = 1_0 \\ 0, \text{ otherwise} \end{cases}$$

Whenever the assumptions and restrictions given above are met, we can uniquely specify any plan $S \varepsilon S_2$ by,

$I_n = 5$: Inspect, at random, one item from the block. If it is defective replace it with a non-defective and in addition inspect all items of the block that follow the defective item by the inspection station. Among these items allow the non-defectives to pass. (This inspection-disposition procedure assumes that the block of items passes by the inspector in single file order.)

$I_n = 6$: Inspect, at random, one item from the block. If it is defective replace it with a non-defective and in addition select, randomly, a second sample of size min $\{a-1, \overline{B}_n - 1\}$, $a>1$. Allow all uninspected items to pass by the inspection station and allow the non-defectives in the random sample to pass by also.

The inspection rule for any plan $S \varepsilon S_2$ can now be defined by the sets $\{\overline{B}_n : n \varepsilon N(S)\}$ and $\{I_n : n \varepsilon N(S)\}$. An inspection point $t(t=0,1,\ldots)$ for block sampling is a point in time when the inspector designates the first member of the next block to be inspected. In general these points will not be equally spaced in time.

Between inspection points t and (t+1), $t=0,1,\ldots,$ and when the present state of a block sampling plan (at point t) is n_u, the application of the plan's inspection rule leads to one of the following results:

(i) if $I_n = 1$, 2 or 3,

$$r = \begin{cases} 0, & \text{if the inspected item is non-defective} \\ 1, & \text{if the inspected item is defective;} \end{cases}$$

(ii) if $I_n = 4$, 5 or 6,

(8) $S = [\{(\bar{B}_n, s_n): n\varepsilon N(S)\}; \{I(n): n\varepsilon N(S)\}; \{J(n): n\varepsilon N(S)\}; \{E(r): r=0,\ldots,\bar{B}_\lambda\}]$.

Thus (4) and (8) define the plans that make up the two sub-classes of the class S_D of all Dodge-type continuous sampling inspection plans.

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